

THE PLURICLOSED FLOW ON NILMANIFOLDS AND TAMED SYMPLECTIC FORMS

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ABSTRACT. We study evolution of (strong Kähler with torsion) SKT structures via the pluriclosed flow on complex nilmanifolds, i.e. on compact quotients of simply connected nilpotent Lie groups by discrete subgroups endowed with an invariant complex structure. Adapting to our case the techniques introduced by Jorge Lauret for studying Ricci flow on homogeneous spaces we show that for SKT Lie algebras the pluriclosed flow is equivalent to a bracket flow and we prove a long time existence result in the nilpotent case. Finally, we introduce a natural flow for evolving tamed symplectic forms on a complex manifold, by considering evolution of symplectic forms via the flow induced by the Bismut Ricci form.

1. INTRODUCTION

Let M be a Hermitian manifold of complex dimension n . If M is non-Kähler, then the Levi-Civita connection is not compatible with the induced $U(n)$ -structure and its role is often replaced by other connections having torsion but preserving the Hermitian structure [9]. Although there is not a canonical choice of a Hermitian connection, the Chern and the Bismut connection seem to have a central role. The Chern connection is defined as the unique Hermitian connection ∇^C for which the $(1,1)$ -component of the torsion tensor vanishes, while the Bismut connection has skew-symmetric torsion [2]. Streets and Tian pointed out in [19] that the operator $g \mapsto S(g)$ defined from the space of Hermitian metrics on a complex manifold (M, J) as

$$S(g)_{i\bar{j}} = g^{\bar{k}r} R_{i\bar{j}r\bar{k}}^C$$

is elliptic, where $R^C = [\nabla^C, \nabla^C] - \nabla^C - \nabla_{[\cdot, \cdot]}^C$ is the curvature of ∇^C . Consequently the standard theory of parabolic equations ensures that the Ricci-type flow

$$(1.1) \quad \begin{cases} \frac{d}{dt}g = S(g) + L(g) \\ g(0) = g_0. \end{cases}$$

has a unique maximal solution defined in an interval $[0, T)$, where L is an arbitrary first order differential operator. Moreover, if we take as $L(g)$ a suitable operator $Q(g)$ depending quadratically on the torsion of ∇^C , the flow

$$(1.2) \quad \begin{cases} \frac{d}{dt}g = S(g) + Q(g) \\ g(0) = g_0 \end{cases}$$

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is the gradient flow of a functional $\mathbb{F} = \mathbb{F}(g)$. The flow (1.2) is called the *Hermitian curvature flow* and preserves both the Kähler and the SKT condition (see [18]). We recall that a Hermitian structure (J, g) is called SKT (strong Kähler with torsion) or *pluriclosed* if its fundamental form $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$ is $\partial\bar{\partial}$ -closed. In the SKT case, (1.2) is equivalent to the so-called pluriclosed flow

$$(1.3) \quad \begin{cases} \frac{d}{dt}\omega = -(\rho^B)^{1,1} \\ \omega(0) = \omega_0 \end{cases}$$

acting on the space of J -compatible non-degenerate real 2-forms, where $(\rho^B)^{1,1}$ denotes the $(1,1)$ -part of the Ricci form ρ^B of the Bismut connection (i.e. the $(1,1)$ -part of the so-called Bismut Ricci form). In the terminology of [18] an SKT structure ω is called static if it satisfies the Einstein-type equation

$$(1.4) \quad r\omega = (\rho^B)^{1,1}$$

where $r \in \mathbb{R}$. Static SKT structures seems to be very rare in complex non-Kähler manifolds, since if ω is static with $r \neq 0$, then $\Omega = \frac{1}{r}\rho^B$ is a symplectic form taming J (i.e. a *Hermitian-symplectic structure* in the terminology of [18]). If a complex surface admits a Hermitian-symplectic structure Ω , then by [16] the Hermitian metric associated to $\Omega^{1,1}$ is *strongly Gauduchon*. Indeed, by [16, Lemma 3.2] a complex manifold (M, J) of complex dimension n carries a strongly Gauduchon metric g if and only if there exists a real d -closed C^∞ $(2n-2)$ -form Ω on M such that its component of type $(n-1, n-1)$ is positive on M .

It is known that every compact complex surface admitting a Hermitian-symplectic structure is actually Kähler (see [18, 15]) and it is still an open problem to find out an example of a compact Hermitian-symplectic manifold not admitting Kähler structures. Some negative results on this context are proved in [6] for nilmanifolds and in [5] for 4-dimensional Lie algebras. In particular, from the results of [6] it turns out that a nilmanifold endowed with an invariant complex structure does not admit Hermitian-symplectic structures unless it is a torus. This last result together with a theorem of [4] implies that complex nilmanifolds cannot admit static SKT structures unless they are tori.

The present paper is divided in two parts. In the first one we investigate the behaviour of solutions of (1.3) on Lie algebras. In particular we prove the following

Theorem 1.1. *Let $(M = G \backslash \Gamma, J, \omega_0)$ be a nilmanifold endowed with an invariant SKT structure. Then the solution $\omega(t)$ to the pluriclosed flow (1.3) is defined for every $t \in (-\epsilon, \infty)$, where ϵ is a suitable positive real number.*

A key ingredient in the proof of this theorem is a trick introduced by Lauret in [13] for studying the Ricci flow on homogeneous spaces evolving the brackets instead of the Riemannian metrics.

In the second part of the paper we introduce a natural flow for evolving taming symplectic forms on a complex manifold (M, J) . Given a symplectic form Ω_0 taming J , we consider the flow

$$(1.5) \quad \begin{cases} \frac{d}{dt}\Omega = -\rho^B(\omega), \\ \Omega(0) = \Omega_0 \end{cases}$$

where $\rho^B(\omega)$ is the Bismut Ricci form of the Hermitian metric associated to $\omega = \Omega^{1,1}$. For such a flow we prove a short time existence result and a stability theorem involving Kähler-Einstein metrics.

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2. PRELIMINARIES ON SKT METRICS

Let (M, g, J) be a Hermitian manifold with fundamental form ω . The form ω and the Riemannian metric g are related as

$$\omega(\cdot, \cdot) = g(\cdot, J\cdot).$$

We denote by ∇^B the Bismut connection of (g, J) . This connection was introduced by Bismut in [2] and it is the unique Hermitian connection (i.e. $\nabla^B J = 0$, $\nabla^B g = 0$) such that

$$(2.1) \quad c(X, Y, Y) := g(X, T^B(Y, Z))$$

is a 3-form, where by

$$T^B(X, Y) = \nabla_X^B Y - \nabla_Y^B X - [X, Y]$$

we denote the torsion of ∇^B . This connection induces the curvature tensor

$$R^B(X, Y) := [\nabla_X^B, \nabla_Y^B] - \nabla_{[X, Y]}^B,$$

the Ricci tensor and the Ricci form given respectively by

$$ric^B(X, Y) = g^{kr} R^B(e_k, X, Y, e_r), \quad \rho^B(X, Y) = \frac{1}{2} g^{kr} g(R^B(X, Y)e_k, J e_r),$$

with $\{e_k\}$ an arbitrary local frame. In complex notation we can alternatively write

$$ric^B(X, Y) = -ig^{\bar{k}k} R^B(Z_k, X, Y, Z_{\bar{k}}), \quad \rho^B(X, Y) = -ig^{\bar{k}k} g(R^B(X, Y)Z_k, Z_{\bar{k}}),$$

where $\{Z_k = \frac{1}{2}(e_k - iJ e_k)\}$ is a local $(1, 0)$ -frame and $Z_{\bar{k}} = \frac{1}{2}(e_k + iJ e_k)$.

For a Hermitian manifold (M, J) the Ricci tensor of ∇^B and the usual Ricci tensor are related by the following formula

$$(2.2) \quad ric^B(X, Y) = ric^g(X, Y) - \frac{1}{2}(d^*c)(X, Y) - \frac{1}{4} \sum_{i=1}^{2n} g(T(X, e_i), T(Y, e_i)),$$

(see [12]), where d^* is the co-differential operator of g , while ρ^B is related to the Ricci form ρ^C of the Chern connection by

$$(2.3) \quad \rho^B = \rho^C - d^*\omega.$$

We recall the following

Definition 2.1. A Hermitian metric g on a complex manifold (M, J) is called SKT (strong Kähler with torsion) or pluriclosed if the torsion 3-form c is closed or, equivalently, if its associated fundamental form ω satisfies $\partial\bar{\partial}\omega = 0$.

For a complex surface, a SKT metric is called *standard* in the terminology of Gauduchon [10]. In the conformal class of any given Hermitian metric on a compact complex manifold there exists always a standard metric. But this property is not anymore true in higher dimensions for SKT metrics.

For real Lie algebras admitting left-invariant SKT metrics there are some classification results in dimensions four, six and eight. More precisely, 6-dimensional

(resp. 8-dimensional) SKT nilpotent Lie algebras have been classified in [7] (resp. in [6] and for a particular class in [17]) and a classification of 4-dimensional SKT solvable Lie algebras has been obtained in [14].

General results are known for nilmanifolds, i.e. for compact quotients of simply connected nilpotent Lie groups G by discrete subgroups Γ . Indeed, in [11] it has been shown that if $(M = G/\Gamma, J)$ is a nilmanifold (not a torus) endowed with an invariant complex structure J and a SKT metric g compatible with J , then the nilpotent Lie group G must be 2-step nilpotent and M is a total space of a principal holomorphic torus bundle over a torus.

3. PLURICLOSED FLOW ON LIE ALGEBRAS

Let G be a Lie group with a left-invariant SKT structure (g_0, J) and let Γ be a co-compact lattice in G . We are interested on studying solutions to (1.3) on the compact manifold $M = G/\Gamma$ endowed with the invariant SKT structure induced by (g_0, J) . Since the pluriclosed flow (1.3) is invariant by biholomorphisms of the complex manifold (M, J) , when ω_0 is invariant, the solution $\omega(t)$ to (1.3) is invariant for every t . Therefore the PDE system (1.3) on $M = G/\Gamma$ is equivalent to an ODE system on the Lie algebra \mathfrak{g} of G .

Let (\mathfrak{g}, μ) be a Lie algebra endowed with an SKT structure (g, J) , where μ denotes the Lie bracket on \mathfrak{g} . By an SKT structure on a Lie algebra we means a pair (g, J) , where J is a complex structure, satisfying the integrability condition

$$\mu(JX, JY) = J\mu(JX, Y) + J\mu(X, JY) + \mu(X, Y),$$

for every $X, Y \in \mathfrak{g}$ and g is an inner product compatible with J and such that $dc = 0$, where c is defined by (2.1). In order to write down a formula for the Bismut Ricci form ρ^B in this algebraic context, we fix an arbitrary $(1, 0)$ frame $\{Z_r\}$ of (\mathfrak{g}, J) with dual frame $\{\zeta^k\}$. Following the approach of [20] we can write

$$\rho^B = d\eta$$

where η is the real 1-form

$$(3.1) \quad \eta(X) = \Im \left\{ g^{\bar{k}r} g(\mu(X - iJX, Z_r), Z_{\bar{k}}) \right\} + i g^{\bar{k}r} g(\mu(Z_r, Z_{\bar{k}}), X).$$

In complex notation we have

$$\eta = \eta_a \zeta^a + \eta_{\bar{b}} \zeta^{\bar{b}},$$

where

$$(3.2) \quad \eta_a = -i g^{\bar{k}r} g(\mu(Z_a, Z_r), Z_{\bar{k}}) + i g^{\bar{k}r} g(\mu(Z_r, Z_{\bar{k}}), Z_a), \quad \eta_{\bar{b}} = \overline{\eta_b}.$$

Formula (3.2) can be rewritten in terms of the Lie bracket components μ_{ij}^k as

$$\eta_a = -i \mu_{ar}^r + i g^{\bar{k}r} \mu_{r\bar{k}}^{\bar{l}} g_{a\bar{l}}, \quad \eta_{\bar{b}} = \overline{\eta_b}.$$

Therefore, in complex notation we obtain

$$\rho^B = -i \rho_{i\bar{j}}^B \zeta^i \wedge \zeta^{\bar{j}} - \frac{i}{2} \rho_{hk}^B \zeta^h \wedge \zeta^{\bar{k}} - \frac{i}{2} \rho_{\bar{l}\bar{m}}^B \zeta^{\bar{l}} \wedge \zeta^{\bar{m}}, \quad \rho_{i\bar{j}}^B = g^{\bar{s}k} R_{i\bar{j}s\bar{k}},$$

where

$$\begin{aligned}\rho_{i\bar{j}}^B &= -i\eta(\mu(Z_i, Z_{\bar{j}})) = -i\mu_{i\bar{j}}^a \eta_a - i\mu_{i\bar{j}}^{\bar{b}} \eta_{\bar{b}} \\ &= -\mu_{i\bar{j}}^a \mu_{a\bar{r}}^r + \mu_{i\bar{j}}^a g^{\bar{k}r} \mu_{r\bar{k}}^{\bar{l}} g_{a\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} \mu_{\bar{b}\bar{r}}^{\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} g^{k\bar{r}} \mu_{k\bar{r}}^l g_{l\bar{b}},\end{aligned}$$

i.e.,

$$(3.3) \quad \rho_{i\bar{j}}^B = -\mu_{i\bar{j}}^a \mu_{a\bar{r}}^r + \mu_{i\bar{j}}^a g^{\bar{k}r} \mu_{r\bar{k}}^{\bar{l}} g_{a\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} \mu_{\bar{b}\bar{r}}^{\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} g^{k\bar{r}} \mu_{k\bar{r}}^l g_{l\bar{b}}.$$

In the same way

$$(3.4) \quad \rho_{i\bar{j}}^B = -\mu_{i\bar{j}}^a \mu_{a\bar{r}}^r + \mu_{i\bar{j}}^a g^{\bar{k}r} \mu_{r\bar{k}}^{\bar{l}} g_{a\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} \mu_{\bar{b}\bar{r}}^{\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} g^{k\bar{r}} \mu_{k\bar{r}}^l g_{l\bar{b}}$$

and (1.2) writes as

$$(3.5) \quad \begin{cases} \frac{d}{dt} g_{i\bar{j}} = -\mu_{i\bar{j}}^a \mu_{a\bar{r}}^r + \mu_{i\bar{j}}^a g^{\bar{k}r} \mu_{r\bar{k}}^{\bar{l}} g_{a\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} \mu_{\bar{b}\bar{r}}^{\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} g^{k\bar{r}} \mu_{k\bar{r}}^l g_{l\bar{b}} \\ g_{i\bar{j}}(0) = (g_0)_{i\bar{j}}. \end{cases}$$

Note that when $\{Z_r\}$ is a unitary frame we have

$$(3.6) \quad \rho_{i\bar{j}}^B = -\mu_{i\bar{j}}^a \mu_{a\bar{r}}^r + \mu_{i\bar{j}}^a \mu_{r\bar{r}}^{\bar{a}} + \mu_{i\bar{j}}^{\bar{b}} \mu_{\bar{b}\bar{r}}^{\bar{l}} + \mu_{i\bar{j}}^{\bar{l}} \mu_{r\bar{r}}^l$$

$$(3.7) \quad \rho_{i\bar{j}}^B = -\mu_{i\bar{j}}^a \mu_{a\bar{r}}^r + \mu_{i\bar{j}}^l \mu_{k\bar{k}}^{\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} \mu_{\bar{b}\bar{r}}^{\bar{l}} + \mu_{i\bar{j}}^{\bar{l}} \mu_{k\bar{k}}^l,$$

where the repeated indexes are summed.

If the Lie algebra \mathfrak{g} is 2-step nilpotent, i.e. if

$$\mu(\mu(X, Y), Z) = 0$$

for every $X, Y, Z \in \mathfrak{g}$, then formulas (3.3) and (3.4) reduce to

$$(3.8) \quad \rho_{i\bar{j}}^B = \mu_{i\bar{j}}^a g^{r\bar{k}} \mu_{r\bar{k}}^{\bar{l}} g_{a\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} g^{k\bar{r}} \mu_{k\bar{r}}^l g_{l\bar{b}}.$$

$$(3.9) \quad \rho_{i\bar{j}}^B = \mu_{i\bar{j}}^a g^{r\bar{k}} \mu_{r\bar{k}}^{\bar{l}} g_{a\bar{l}} + \mu_{i\bar{j}}^{\bar{b}} g^{k\bar{r}} \mu_{k\bar{r}}^l g_{l\bar{b}}.$$

giving the suitable expression

$$(3.10) \quad \rho^B(X, Y) = -ig^{\bar{k}r} g([X, Y], [Z_r, Z_{\bar{k}}]),$$

for every $X, Y \in \mathfrak{g}$.

Remark 3.1. We observe that previous computations hold also in the non-SKT case.

We will now show two examples of SKT Lie algebras in dimension 4 for which the solution $\omega(t)$ of the pluriclosed flow (1.3) is defined for every $t \in (-\epsilon, \infty)$, where ϵ is a suitable positive real number. The first example is nilpotent and we will show in the next section that this happens for every SKT nilpotent Lie algebra. The second one is solvable and admits a generalized Kähler structure [1, 8].

Example 3.2. In dimension 4 the unique nilpotent SKT Lie algebra up to isomorphisms is $\mathfrak{h}_3 \oplus \mathbb{R}$, where \mathfrak{h}_3 is the Lie algebra of the 3-dimensional real Heisenberg Lie group $H_3(\mathbb{R})$ given by

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\}.$$

The compact quotient of the corresponding simply-connected $H_3(\mathbb{R}) \times \mathbb{R}$ by the lattice $\Gamma \times \mathbb{Z}$, where Γ is the lattice in $H_3(\mathbb{R})$ whose elements are matrices with

integer entries, is the so-called Kodaira-Thurston surface. The Lie algebra $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbb{R}$ has structure equations $(0, 0, 0, 12)$, where by this notation we mean that there exists a basis of 1-forms $\{e^i\}$ such that

$$de^i = 0, i = 1, 2, 3, \quad de^4 = e^1 \wedge e^2.$$

Let J be the complex structure on \mathfrak{g} given by

$$Je_1 = -e_2, \quad Je_3 = -e_4.$$

Then

$$Z_1 = \frac{1}{2}(e_1 + ie_2), \quad Z_2 = \frac{1}{2}(e_3 + ie_4)$$

is a complex basis of type $(1, 0)$ of (\mathfrak{g}, J) . Let $\{\zeta^1, \zeta^2\}$ be its dual frame. Every Hermitian inner product g on (\mathfrak{g}, J) can be written as

$$g = x\zeta^1\zeta^{\bar{1}} + y\zeta^2\zeta^{\bar{2}} + z\zeta^1\zeta^{\bar{2}} + \bar{z}\zeta^2\zeta^{\bar{1}}.$$

where $x, y \in \mathbb{R}$, $z \in \mathbb{C}$ satisfy $xy - |z|^2 > 0$ and it is SKT. Since

$$\mu(Z_1, Z_{\bar{1}}) = -\frac{1}{2}(Z_2 - Z_{\bar{2}})$$

is the only non-vanishing bracket we have

$$\rho^B = -i\rho_{1\bar{1}}^B \zeta^{1\bar{1}}$$

where

$$\rho_{1\bar{1}}^B = -i\eta([Z_1, Z_{\bar{1}}]) = i\frac{1}{2}(\eta_2 - \eta_{\bar{2}}) = -\Im \eta_2.$$

A direct computation yields

$$\eta_2 = i\frac{y^2}{2(xy - |z|^2)}$$

and

$$\rho_{1\bar{1}}^B = -\frac{y^2}{2(xy - |z|^2)}.$$

Therefore in this case system (3.5) reduces to

$$(3.11) \quad \begin{cases} \dot{x} = \frac{y^2}{2(xy - |z|^2)} \\ y \equiv y_0, \quad z \equiv z_0 \\ x(0) = x_0, \end{cases}$$

and the solution to (4.1) with

$$\omega_0 = ix_0\zeta^{1\bar{1}} + iy_0\zeta^{2\bar{2}} + iz_0\zeta^{1\bar{2}} - iz_0\zeta^{2\bar{1}}$$

is

$$\omega(t) = ix(t)\zeta^{1\bar{1}} + iy_0\zeta^{2\bar{2}} + iz_0\zeta^{1\bar{2}} - iz_0\zeta^{2\bar{1}}$$

where

$$x(t) = \frac{1}{y_0} \left(\sqrt{y_0^2 t + (x_0 y_0 - |z_0|^2)^2 + |z_0|^2} \right).$$

For instance if we start from the standard SKT structure

$$\omega_0 = \zeta^{1\bar{1}} + \zeta^{2\bar{2}}$$

we get

$$\omega(t) = \sqrt{1+t} \zeta^{1\bar{1}} + \zeta^{2\bar{2}}.$$

Example 3.3. Consider the solvable Lie algebra with structure equations

$$\begin{cases} de^1 = ae^{14} + be^{24}, \\ de^2 = -be^{14} + ae^{24}, \\ de^3 = -2ae^{34} \\ de^4 = 0, \end{cases} \quad a, b \in \mathbb{R} - \{0\},$$

endowed with the complex structure given by

$$Je_1 = e_2, \quad Je_3 = e_4.$$

A compact quotient of the corresponding simply-connected Lie group by a lattice is a Inoue surface of type S^0 (see [11]). Let $\{Z_1, Z_2\}$ be the $(1, 0)$ -frame

$$Z_1 = \frac{1}{2}(e_1 - ie_2), \quad Z_2 = \frac{1}{2}(e_3 - ie_4);$$

then a direct computation yields

$$\begin{aligned} \mu(Z_1, Z_{\bar{1}}) &= 0, \quad \mu(Z_1, Z_2) = \lambda Z_1, \quad \mu(Z_{\bar{1}}, Z_{\bar{2}}) = \bar{\lambda} Z_{\bar{1}}, \\ \mu(Z_1, Z_{\bar{2}}) &= -\lambda Z_1, \quad \mu(Z_{\bar{1}}, Z_2) = -\bar{\lambda} Z_{\bar{1}}, \quad \mu(Z_2, Z_{\bar{2}}) = ai(Z_2 + Z_{\bar{2}}) \end{aligned}$$

where

$$\lambda = \frac{b + ia}{2}.$$

Consider the $(1, 0)$ -coframe

$$\zeta^1 = e^1 + ie^2, \quad \zeta^2 = e^3 + ie^4$$

dual to $\{Z_1, Z_2\}$. Then

$$d\zeta^1 = -\lambda(\zeta^{12} - \zeta^{1\bar{2}}), \quad d\zeta^2 = -ai\zeta^{2\bar{2}}.$$

Let g be an arbitrary J -Hermitian metric on \mathfrak{g} . We can write

$$g = x\zeta^1\zeta^{\bar{1}} + y\zeta^2\zeta^{\bar{2}} + z\zeta^1\zeta^{\bar{2}} + \bar{z}\zeta^2\zeta^{\bar{1}}$$

where $x, y \in \mathbb{R}_+$, $z \in \mathbb{C}$ satisfy

$$xy - |z|^2 > 0.$$

The fundamental form of g is

$$\omega = -ix\zeta^{1\bar{1}} - iy\zeta^{2\bar{2}} - iz\zeta^{1\bar{2}} - i\bar{z}\zeta^{2\bar{1}}.$$

Let $\rho^B = d\eta$ be the Bismut form of g . A direct computation yields

$$\eta_1 = -\frac{xz}{xy - |z|^2}(i\bar{\lambda} + a) = -\frac{3a + ib}{2} \frac{xz}{xy - |z|^2}$$

and

$$\eta_2 = i\lambda - \frac{i\bar{\lambda}|z|^2 + axy}{xy - |z|^2} - \frac{i|z|^2(\lambda + \bar{\lambda}) + xy(a - i\lambda)}{xy - |z|^2}.$$

In matrix notation we have

$$(g_{i\bar{j}}) = \begin{pmatrix} x & z \\ \bar{z} & y \end{pmatrix}$$

and

$$(g^{\bar{j}i}) = \frac{1}{xy - |z|^2} \begin{pmatrix} y & -z \\ -\bar{z} & x \end{pmatrix}.$$

A direct computation yields

$$(3.12) \quad \rho_{1\bar{1}}^B = 0,$$

$$(3.13) \quad \rho_{1\bar{2}}^B = \lambda\theta_1 = \frac{1}{4}(3a^2 + b^2 - 2iab) \frac{xz}{xy - |z|^2},$$

$$(3.14) \quad \rho_{2\bar{2}}^B = -\frac{3a^2xy}{xy - |z|^2}$$

and the ODEs induced by (3.5) are

$$(3.15) \quad \begin{cases} x = \text{const}, \\ \dot{z} = -\frac{1}{4}(3a^2 + b^2 - 2iab) \frac{xz}{xy - |z|^2}, \\ \dot{y} = \frac{3a^2xy}{xy - |z|^2}. \end{cases}$$

In particular if we consider as starting SKT structure

$$\omega_0 = i\zeta^{1\bar{1}} + i\zeta^{2\bar{2}},$$

then the system (3.15) has solutions

$$\begin{cases} x \equiv 1 \\ z \equiv 0 \\ y(t) = 3a^2t \end{cases}$$

defined for every t and

$$\omega(t) = i\zeta^{1\bar{1}} + i3a^2t\zeta^{2\bar{2}}.$$

4. THE PLURICLOSED FLOW AS BRACKET FLOW

We regard the pluriclosed flow (1.3) on Lie algebras as a bracket flow on \mathbb{R}^{2n} working as in [13]. The idea consists on studying evolution of brackets instead of symplectic forms. We briefly describe the clever trick of [13] adapted to our setting:

Let $(\mathfrak{g}, \mu_0, J, g_0, \omega_0)$ be a almost Hermitian Lie algebra. Then $(\mathfrak{g}, \mu_0, J, \omega_0)$ can be thought as \mathbb{R}^{2n} equipped with the standard Hermitian structure $(J_0, \omega_0, \langle \cdot, \cdot \rangle)$ and a bracket μ_0 . Consider in this setting

$$(4.1) \quad \begin{cases} \frac{d}{dt}\omega = -(\rho^B)^{1,1}(\omega) \\ \omega(0) = \omega_0 \end{cases}$$

where $\rho^B(\omega)$ is computed with respect to ω and μ_0 using formulae (3.6), i.e.,

$$(4.2) \quad \rho^B(\omega)(X, Y) = i \sum_{r=1}^n \left(g(\mu_0(\mu_0(X, Y), Z_r), Z_{\bar{r}}) - g(\mu_0(Z_r, Z_{\bar{r}}), \mu_0(X, Y)) \right)$$

g is the inner product induced by (ω, J_0) and $\{Z_r\}$ is a unitary frame.

Let

$$V := \Lambda^2(\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$$

be the space of skew-symmetric 2-forms on \mathbb{R}^{2n} taking values in \mathbb{R}^{2n} and let

$$(4.3) \quad \mathcal{A} := \{\mu \in V : \mu \text{ satisfies the Jacoby identity and } N_\mu = 0\}$$

where N_μ is the Nijenhuis tensor

$$N_\mu(X, Y) := \mu(J_0X, J_0Y) - J_0\mu(J_0X, Y) - J_0\mu(X, J_0Y) - \mu(X, Y).$$

\mathcal{A} can be regarded as the space of all $2n$ -dimensional Lie algebras equipped with a complex structure. Given a form $\omega \in \Lambda^2(\mathbb{R}^{2n})^*$ compatible with J_0 , there exists a (non-unique) map $h \in \text{GL}(n, \mathbb{C})$ such that

$$(4.4) \quad \omega(\cdot, \cdot) = \omega_0(h\cdot, h\cdot).$$

For every $h \in \text{GL}(n, \mathbb{C})$ satisfying (4.4), we can define a new bracket $\mu \in \mathcal{A}$ using the natural relation

$$\mu := h \cdot \mu_0$$

where

$$h \cdot \mu_0(X, Y) = h \mu_0(h^{-1}X, h^{-1}Y).$$

Since h belongs to $\text{GL}(n, \mathbb{C})$, $h \cdot \mu \in \mathcal{A}$. Moreover, every $\mu \in \mathcal{A}$ induces a 2-form $\rho_\mu^B \in \Lambda^2(\mathbb{R}^{2n})^*$ defined according to (3.2) as

$$\rho_\mu^B(X, Y) = i \sum_{r=1}^n \left(\langle \mu(\mu(X, Y), Z_r), Z_{\bar{r}} \rangle - \langle \mu(Z_r, Z_{\bar{r}}), \mu(X, Y) \rangle \right),$$

where $\{Z_r\}$ be the standard unitary basis on (\mathbb{R}^{2n}, J_0) . We denote by P_μ the endomorphism corresponding to $(\rho_\mu^B)^{1,1}$ and μ via ω_0 , i.e.

$$(4.5) \quad \omega_0(P_\mu(X), Y) = (\rho_\mu^B)^{1,1}(X, Y).$$

By definition P_μ is ω_0 -symmetric and it commutes with J_0 .

In the same way we denote by $P(\omega)$ the endomorphism corresponding $(\rho^B)^{1,1}(\omega)$ via μ_0 , i.e.

$$\omega(P(\omega)X, Y) = (\rho^B)^{1,1}(\omega).$$

Note that

$$P_{\mu_0} = P(\omega_0),$$

since

$$\rho^B(\omega_0) = \rho_{\mu_0}^B.$$

The following lemma describes as the two endomorphisms P_μ and $P(\omega)$ are related:

Lemma 4.1. *The following formula holds*

$$(4.6) \quad P_\mu = h P(\omega) h^{-1}.$$

Proof. The proof consists of a straightforward computation. We have

$$\omega_0(hP(\omega)h^{-1}X, Y) = \omega_0(hP(\omega)h^{-1}X, hh^{-1}Y) = \omega(P(\omega)h^{-1}X, h^{-1}Y)$$

and formula (4.2) implies

$$\begin{aligned} \omega(P(\omega)h^{-1}X, h^{-1}Y) &= -i \sum_{r=1}^n \left(g(\mu_0(\mu_0(h^{-1}X, h^{-1}Y), Z_r), Z_{\bar{r}}) \right. \\ &\quad \left. - g(\mu_0(Z_r, Z_{\bar{r}}), \mu_0(h^{-1}X, h^{-1}Y)) \right). \end{aligned}$$

$\{Z_r\}$ being a unitary frame with respect to g . Therefore

$$\begin{aligned}
\omega_0(h^{-1}P(\omega)h^{-1}XY) &= \\
& i \sum_{r=1}^n \left(g(\mu_0(h^{-1}\mu(X, Y), Z_r), Z_{\bar{r}}) - g(\mu_0(Z_r, Z_{\bar{r}}), h^{-1}\mu(X, Y)) \right) \\
& i \sum_{r=1}^n \left(g(h^{-1}\mu(\mu(X, Y), hZ_r), h^{-1}hZ_{\bar{r}}) - g(h^{-1}\mu(hZ_r, hZ_{\bar{r}}), h^{-1}\mu(X, Y)) \right) \\
& = i \sum_{r=1}^n \left(\langle \mu(\mu(X, Y), hZ_r), hZ_{\bar{r}} \rangle - \langle (h^{-1}\mu(hZ_r, hZ_{\bar{r}}), \mu(X, Y)) \rangle \right) \\
& = \rho_\mu^B(X, Y),
\end{aligned}$$

as required. \square

We further consider the bracket flow

$$(4.7) \quad \begin{cases} \frac{d}{dt}\mu = \frac{1}{2}\delta_\mu(P_\mu) \\ \mu(0) = \mu_0 \end{cases}$$

where $\delta_\mu: \mathfrak{gl}_n(\mathbb{C}) \rightarrow V_{2n}$ is defined by

$$\delta_\mu(\alpha) = \mu(\alpha, \cdot) + \mu(\cdot, \alpha) - \alpha\mu(\cdot, \cdot).$$

Theorem 4.2. *Let $\omega(t)$ be a solution to (4.1) and let $\mu(t)$ be a solution to (4.7). Then there exists a curve $h = h(t) \in \mathrm{GL}(n, \mathbb{C})$ such that:*

1. $\omega = h^*\omega_0$;
2. $\mu = h \cdot \mu_0$;
3. $\frac{d}{dt}h = -\frac{1}{2}hP(\omega) = -\frac{1}{2}P_\mu h$.

Proof. Let $\omega = \omega(t)$ be a solution to (4.1) and let $h(t)$ be the solution to the linear ODE

$$\begin{cases} \frac{d}{dt}h = -\frac{1}{2}hP(\omega) \\ h_0 = \mathrm{I}. \end{cases}$$

If $\tilde{\omega} = h^*\omega_0$, then

$$\begin{aligned}
\frac{d}{dt}\tilde{\omega}(\cdot, \cdot) &= \omega_0(h' \cdot, h \cdot) + \omega_0(h \cdot, h' \cdot) = -\frac{1}{2}\omega_0(hP(\omega) \cdot, h \cdot) - \frac{1}{2}\omega_0(h \cdot, hP(\omega) \cdot) \\
&= -\frac{1}{2}\tilde{\omega}(P(\omega) \cdot, \cdot) - \frac{1}{2}\tilde{\omega}(\cdot, P(\omega) \cdot)
\end{aligned}$$

Since

$$\frac{d}{dt}\omega(\cdot, \cdot) = -\frac{1}{2}\omega(P(\omega) \cdot, \cdot) - \frac{1}{2}\omega(\cdot, P(\omega) \cdot)$$

ω and $\tilde{\omega}$ solve the same ODE

$$\begin{cases} \frac{d}{dt}\beta(\cdot, \cdot) = -\frac{1}{2}\beta(P(\omega) \cdot, \cdot) - \frac{1}{2}\beta(\cdot, P(\omega) \cdot) \\ \beta(0) = \omega_0 \end{cases}$$

and therefore $\omega = h^*\omega_0$. Let $\lambda = h \cdot \mu_0$. Using lemma (4.1) and $h' = -\frac{1}{2}P_\lambda h$ we obtain

$$\begin{aligned}\lambda'(\cdot, \cdot) &= h'\mu_0(h^{-1}\cdot, h^{-1}\cdot) - h\mu_0(h^{-1}h'h^{-1}\cdot, h^{-1}\cdot) - h\mu_0(h^{-1}\cdot, h^{-1}h'h^{-1}\cdot) \\ &= (h'h^{-1})h\mu_0(h^{-1}\cdot, h^{-1}\cdot) - h\mu_0(h^{-1}(h'h^{-1})\cdot, h^{-1}\cdot) - h\mu_0(h^{-1}\cdot, h^{-1}(h'h^{-1})\cdot) \\ &= -\delta_\lambda(h'h^{-1}) = \frac{1}{2}\delta_\lambda(P_\lambda).\end{aligned}$$

Therefore λ and μ solve the same ODE and consequently they coincide, as required. \square

Remark 4.3. Note that the Bismut scalar form $b(\omega) = g(\rho^B(\omega), \omega)$ reads in terms of bracket as

$$b_\mu := \langle \rho_\mu^B, \omega \rangle = \sum_{r,k} \langle \mu(Z_r, Z_{\bar{r}}), \mu(Z_k, Z_{\bar{k}}) \rangle,$$

i.e.,

$$b_\mu = - \left\| \sum_r \mu(Z_r, Z_{\bar{r}}) \right\|^2.$$

$\{Z_r\}$ being an arbitrary unitary frame.

Lemma 4.4. *The bracket flow (4.7) preserves the center of μ_0 .*

Proof. Consider on $(\mathbb{R}^{2n}, J_0, \omega_0)$ an arbitrary $\mu_0 \in \mathcal{A}$. Let ξ_0 be the center of μ_0 and ξ_0^\perp its orthogonal complement with respect to $\langle \cdot, \cdot \rangle$. Every J_0 -compatible non-degenerate form ω can be decomposed as $\omega = \alpha' + \alpha$ with respect the splitting $\mathfrak{g} = \xi \oplus \xi^\perp$, where α is the restriction of ω to $\xi^\perp \times \xi^\perp$. We can write in particular $\omega_0 = \alpha_0 + \alpha'_0$. Formula (4.2) implies that $\rho^B(X^\xi, \cdot)$ vanishes for every $X^\xi \in \xi$. Therefore the solution $\omega(t)$ to (4.1) can be written as $\omega(t) = \alpha'_0 + \alpha(t)$ and every $h = h(t)$ satisfying conditions 1,2,3 of Theorem 4.2 preserves α'_0 , i.e. $h(t)^*(\alpha'_0) = \alpha'_0$ for every t . There follows $h(t)(\xi_0) = \xi_0$ for any t . The solution $\mu(t)$ to the bracket flow (4.7) is defined in terms of h and μ_0 as $\mu(t) = h(t) \cdot \mu_0$ and for every t the kernel of $\mu(t)$ is $\xi_t = h(t)\xi_0$. Hence $\xi_t \equiv \xi_0$, as required. \square

We describe now as the SKT condition reads in terms of brackets:

Let μ be a bracket in \mathcal{A} . Then μ induces the differential operator

$$d_\mu : \Lambda^r(\mathbb{R}^{2n})^* \rightarrow \Lambda^{r+1}(\mathbb{R}^{2n})^*$$

defined in terms of μ as

$$d_\mu \gamma(X_0, X_1, \dots, X_r) = \sum_{0 \leq i < j \leq r} (-1)^{i+j} \gamma(\mu(X_i, X_j), X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r).$$

Furthermore, the complex extension of d_μ splits with respect to J_0 as $d_\mu = \partial_\mu + \bar{\partial}_\mu$. We denote by $\partial, \bar{\partial}$ the usual differential complex operator on (\mathbb{R}^{2n}, J_0) . Every SKT Lie algebra can be seen as a Hermitian Lie algebra $(\mathbb{R}^{2n}, J_0, \langle \cdot, \cdot \rangle, \mu)$ whose fundamental form ω_0 satisfies

$$\bar{\partial}_\mu \partial_\mu \omega_0 = 0.$$

This motivates the following

Definition 4.5. A bracket $\mu \in \mathcal{A}$ is *SKT* if

$$(4.8) \quad \bar{\partial}_\mu \partial_\mu \omega_0 = 0.$$

The identity (4.8) is an algebraic equation in μ and, therefore, the set of SKT brackets gives an algebraic subspace of \mathcal{A} .

5. LONG TIME EXISTENCE FOR NILMANIFOLDS

In this subsection we focus on SKT structures on nilpotent Lie algebras proving Theorem 1.1.

The following two results will be important in the sequel

Theorem 5.1. ([6]) *Let $(\mathfrak{g}, \mu, J, \omega)$ be an SKT nilpotent Lie algebra. Then \mathfrak{g} is 2-step and J preserves the center ξ of \mathfrak{g} .*

Theorem 5.2. ([20]) *For a 2-step nilpotent almost Hermitian Lie algebra, the Chern form ρ^C is always vanishing.*

Therefore in the SKT nilpotent case we have to handle 2-step nilpotent Lie algebras. Using the general formula (2.3)

$$\rho^B = \rho^C - d^*\omega$$

we have that the pluriclosed flow reduces in this case to

$$(5.1) \quad \begin{cases} \frac{d}{dt}\omega = (dd^*\omega)^{1,1}, \\ \omega(0) = \omega_0. \end{cases}$$

Let us consider then an SKT (2-step) nilpotent Lie algebra $(\mathfrak{g}, \mu, J, \omega)$ with induced metric g . We denote by $\sharp: \mathfrak{g} \rightarrow \mathfrak{g}^*$ the duality induced by the inner product g . Given a vector subspace W of \mathfrak{g} we set $W^\sharp := \sharp(W)$ and we denote by W^\perp the orthogonal complement of W with respect to g . Finally we denote by $\theta = -Jd^*\omega$ the Lee form of (J, ω) . We have the following

Proposition 5.3. *The Lee form θ of a nilpotent SKT Lie algebra $(\mathfrak{g}, \mu, J, g)$ belongs to $(J\mathfrak{g}^1)^\sharp \subseteq \xi^\sharp$.*

Proof. By definition of θ we have

$$g(J\theta, \alpha) = g(\omega, d\alpha),$$

for every $\alpha \in \mathfrak{g}^*$. Note that if $\alpha \in ((\mathfrak{g}^1)^\perp)^\sharp$, then $d\alpha = 0$. Therefore, $J\theta \in (\mathfrak{g}^1)^\sharp$, i.e. $\theta \in (J\mathfrak{g}^1)^\sharp$. On the other hand, since \mathfrak{g} is 2-step nilpotent and the center ξ is J -invariant, it follows that $J\mathfrak{g}^1 \subseteq \xi$, i.e. that $\theta \in \xi^\sharp$. \square

Lemma 5.4. *For a nilpotent SKT Lie algebra $(\mathfrak{g}, \mu, J, g)$ the $(1, 1)$ -part $(ric^B)^{1,1}$ of the Ricci tensor of the Bismut connection is symmetric and it is related to ρ^B by*

$$(\rho^B)^{1,1}(X, Y) = (ric^B)^{1,1}(X, JY),$$

for every $X, Y \in \mathfrak{g}$.

Proof. We can write

$$\mathfrak{g} = \xi \oplus \xi^\perp,$$

where ξ is the center of \mathfrak{g} . The 2-step condition implies

$$[\xi^\perp, \xi^\perp] \subseteq \xi.$$

Every $X \in \mathfrak{g}$ can be written accordingly as

$$X = X^\xi + X^\perp,$$

where $X^\xi \in \xi$ and $X^\perp \in \xi^\perp$. By [4, Lemma 2.1] we have that

$$\nabla_{X^\xi}^B Y^\xi = 0, \quad \nabla_{X^\xi}^B Y^\perp \in \xi^\perp, \quad \nabla_{X^\perp}^B Y^\xi \in \xi^\perp$$

and

$$\nabla_{X^\perp}^B Y^\perp = \frac{1}{2}([X^\perp, Y^\perp] - [JX^\perp, JY^\perp]) \in \xi.$$

Therefore

$$\theta(\nabla_X^B Y) = \frac{1}{2}\theta([X^\perp, Y^\perp] - [JX^\perp, JY^\perp]).$$

Using the integrability of J we get

$$\begin{aligned} \theta(\nabla_X^B Y) &= \frac{1}{2}\theta(-J[JX^\perp, Y^\perp] - J[X^\perp, JY^\perp]) \\ &= \frac{1}{2}(J\theta)([JX^\perp, Y^\perp] + [X^\perp, JY^\perp]) \\ &= -\frac{1}{2}d(J\theta)(JX^\perp, Y^\perp) - \frac{1}{2}d(J\theta)(X^\perp, JY^\perp). \end{aligned}$$

Taking into account that $\rho^B = \rho^C + d(J\theta)$ and that ρ^C in our cases vanishes, we have

$$\theta(\nabla_X^B Y) = -\frac{1}{2}\rho^B(JX^\perp, Y^\perp) - \frac{1}{2}\rho^B(X^\perp, JY^\perp).$$

Therefore

$$(\nabla_X^B \theta)(JY) = -\theta(\nabla_X^B JY) = \frac{1}{2}\rho^B(JX^\perp, JY^\perp) - \frac{1}{2}\rho^B(X^\perp, Y^\perp) = -(\rho^B)^{2,0+0,2}(X^\perp, Y^\perp).$$

Using

$$\rho^B(X, Y) = \text{ric}^B(X, JY) + (\nabla_X^B \theta)(JY) = \text{ric}^B(X, JY) - (\rho^B)^{2,0+0,2}(X^\perp, Y^\perp),$$

we get

$$(\rho^B)^{1,1}(X, Y) = \frac{1}{2}[\text{ric}^B(X, JY) - \text{ric}^B(JX, Y)] = (\text{ric}^B)^{1,1}(X, JY),$$

as required. \square

Theorem 5.5. *For a nilpotent SKT Lie algebra $(\mathfrak{g}, \mu, J, g)$ we have*

$$(\text{ric}^B)^{1,1}(X, Y) = 2(\text{ric}^g)^{1,1}(X^\perp, Y^\perp),$$

for every $X, Y \in \mathfrak{g}$.

Proof. Using formula (3.10) and Lemma 5.4 we have that

$$(\text{ric}^B)^{1,1}(X, Y) = (\text{ric}^B)^{1,1}(X^\perp, Y^\perp).$$

In view of formula (2.2) the two Ricci tensors ric^B and ric^g are related by

$$\text{ric}^B(X, Y) = \text{ric}^g(X, Y) - \frac{1}{2}(d^*c)(X, Y) - \frac{1}{4}\sum_{i=1}^{2n} g(T^B(X, e_i), T^B(Y, e_i)),$$

where c and T torsion 3-form and the torsion tensor of the Bismut connection.

Since d^*c is a 2-form, the previous Lemma 5.4 implies

$$(d^*c)^{1,1} = 0$$

and therefore we get

$$(\text{ric}^B)^{1,1}(X, Y) = (\text{ric}^g)^{1,1}(X, Y) - \frac{1}{8}\sum_{i=1}^{2n} [g(T^B(X, e_i), T^B(Y, e_i)) + g(T^B(JX, e_i), T^B(JY, e_i))].$$

Hence the claim consists on showing that

$$\frac{1}{8} \sum_{i=1}^{2n} [g(T^B(X^\perp, e_i), T^B(Y^\perp, e_i)) + g(T^B(JX^\perp, e_i), T^B(JY^\perp, e_i))] = -2(ric^g)^{1,1}(X^\perp, Y^\perp).$$

Let $S: \xi^\perp \rightarrow \xi^\perp$ is the symmetric operator defined by the relation

$$g(S(X^\perp), Y^\perp) = ric^g(X^\perp, Y^\perp).$$

By [3] we have that S can be written as

$$S = \frac{1}{2} \sum_{i=1}^{2p} \iota(z_i)^2,$$

where $\{z_1, \dots, z_{2p}\}$ is an orthonormal basis of ξ and the skew-symmetric map $\iota(Z): \xi^\perp \rightarrow \xi^\perp$ is defined by

$$g(\iota(Z)X, Y) = g([X, Y], Z), \quad X, Y \in \xi^\perp,$$

for $Z \in \xi$. Equivalently $\iota(Z)(X) = -(ad_X)^*(Z)$, for every $X \in \xi^\perp$, where $(ad_X)^*$ is the adjoint of ad_X with respect to the inner product g . In particular S is negative definite on ξ^\perp . There follows

$$ric^g(X^\perp, Y^\perp) = \frac{1}{2} \sum_{i=1}^{2p} g(\iota(z_i)^2(X^\perp), Y^\perp) = -\frac{1}{2} \sum_{i=1}^{2p} g(\iota(z_i)X^\perp, \iota(z_i)Y^\perp),$$

where $\{z_1, \dots, z_{2p}\}$ is an orthonormal basis of ξ . By using that

$$\begin{aligned} g(\nabla_{X^\xi}^B Y^\perp, Z) &= -\frac{1}{2}g([Y^\perp, Z] + [JY^\perp, JZ], X^\xi), \\ g(\nabla_{Y^\perp}^B X^\xi, Z) &= -\frac{1}{2}g([Y^\perp, Z] - [JY^\perp, JZ], X^\xi), \end{aligned}$$

we have that, for every $z_i \in \xi$,

$$\begin{aligned} g(T^B(X^\perp, z_i), T^B(Y^\perp, z_i)) &= g([JX^\perp, J(T^B(Y^\perp, z_i))], z_i) = g(\iota(z_i)(JX^\perp), JT^B(Y^\perp, z_i)) \\ &= -g(J\iota(z_i)(JX^\perp), T^B(Y^\perp, z_i)) = -g([JY^\perp, J^2\iota(z_i)(JX)], z_i) \\ &= g(\iota(z_i)(JY^\perp), \iota(z_i)(JX^\perp)). \end{aligned}$$

Therefore

$$(5.2) \quad \sum_{i=1}^{2p} g(T^B(X^\perp, z_i), T^B(Y^\perp, z_i)) = -2ric^g(JX^\perp, JY^\perp).$$

On the other hand, if $v_i \in \xi^\perp$, then

$$g(T^B(X^\perp, v_i), T^B(Y^\perp, v_i)) = g([Jv_i, JX^\perp], [Jv_i, JY^\perp]).$$

By Section 2 in [3] for a metric 2-step nilpotent Lie algebra one has

$$R^g(X^\perp, Y^\perp)X^\perp = \frac{3}{4}\iota([X^\perp, Y^\perp])X^\perp,$$

for every $X^\perp, Y^\perp \in \xi^\perp$ and consequently

$$g(R^g(X^\perp, Y^\perp)X^\perp, W) = \frac{3}{4}g(\iota([X^\perp, Y^\perp])X^\perp, W) = \frac{3}{4}g([X^\perp, Y^\perp], [X^\perp, W])$$

for every $W \in \mathfrak{g}$. As a consequence, for every $v_i \in \xi^\perp$ we have

$$g(T^B(X^\perp, v_i), T^B(Y^\perp, v_i)) = \frac{4}{3}g(R^g(Jv_i, JX^\perp)Jv_i, JY^\perp).$$

Moreover, by [3, pag. 622]

$$\sum_{i=1}^{2n-p} g(R^g(v_i, X^\perp)Y^\perp, v_i) = \frac{3}{4} \sum_{k=1}^{2p} g(\iota(z_k)^2 X^\perp, Y^\perp).$$

Therefore

$$\sum_{i=1}^{2n-p} g(T^B(X^\perp, v_i), T^B(Y^\perp, v_i)) = - \sum_{k=1}^{2p} g(\iota(Jz_k)^2 JX^\perp, JY^\perp) = -2ric^g(JX^\perp, JY^\perp).$$

In this way

$$(ric^B)^{1,1}(X^\perp, Y^\perp) = 2(ric^g)^{1,1}(X^\perp, Y^\perp).$$

□

Remark 5.6. By [3] for a metric 2-step Lie algebra (\mathfrak{g}, μ, g) one has

1. $ric^g(X, Z) = 0$ for all $X \in \xi$ and $Z \in \xi^\perp$.
2. $ric^g(Z, Z^*) = -\frac{1}{4}\text{tr } \iota(Z)\iota(Z^*)$ for $Z, Z^* \in \xi$. In particular $ric^g(Z, Z) \geq 0$ for all $Z \in \xi$ with equality if and only if $\iota(Z) = 0$.

Moreover, giving a Hermitian structure (g, J) on \mathfrak{g} , for $X \in \xi$ and $Y \in \xi^\perp$ we have

$$g(T^B(X, e_i), T^B(Y, e_i)) = g(\nabla_X^B e_i - \nabla_{e_i}^B X, \nabla_Y^B e_i - \nabla_{e_i}^B Y - [Y, e_i]).$$

If $e_i \in \xi$ then

$$g(T^B(X, e_i), T^B(Y, e_i)) = 0.$$

If $e_i \in \xi^\perp$ we have that $\nabla_X^B e_i - \nabla_{e_i}^B X \in \xi^\perp$ and $\nabla_Y^B e_i - \nabla_{e_i}^B Y - [Y, e_i] \in \xi$. So again $g(T^B(X, e_i), T^B(Y, e_i)) = 0$ and summing up

$$\frac{1}{4} \sum_{i=1}^{2n} g(T^B(X, e_i), T^B(Y, e_i)) = 0$$

for every $X \in \xi$ and $Y^\perp \in \xi^\perp$. There follows that

$$(ric^B)^{1,1}(X, Y^\perp) = 2(ric^g)^{1,1}(X^\perp, Y)$$

for all $X \in \mathfrak{g}$ and $Y \in \xi^\perp$, while

$$(ric^B)^{1,1}_{\xi \times \xi} \neq 2(ric^g)^{1,1}_{\xi \times \xi}.$$

Let us consider now the space \mathcal{N} of all $2n$ -dimensional nilpotent Lie algebras equipped with a complex structure. Such a space can be seen as a subspace of the space \mathcal{A} defined in (4.3). Let μ_0 be an SKT bracket in \mathcal{N} and let $\mu(t)$ be the solution to (4.7) satisfying $\mu(0) = \mu_0$. Then $\mu(t)$ is SKT for every t and we have

$$\frac{d}{dt} \langle \mu, \mu \rangle = 2 \left\langle \frac{d}{dt} \mu, \mu \right\rangle = \langle \delta_\mu(P_\mu), \mu \rangle = -4 \langle P_\mu, Ric_\mu \rangle$$

where $Ric_\mu = -\frac{1}{2} \sum (\text{ad}_\mu e_i)^t \text{ad}_\mu e_i + \frac{1}{4} \sum \text{ad}_\mu e_i (\text{ad}_\mu e_i)^t$ is the usual Ricci operator induced by μ (see [13, Lemma 4.2]). Using that P_μ is of type $(1, 1)$ (i.e. that it commutes with J_0) and that P_μ vanishes along the center of μ

$$\xi_\mu = \{X \in \mathbb{R}^{2n} \text{ s.t. } \mu(X, Y) = 0 \text{ for all } Y \in \mathbb{R}^{2n}\}$$

we have

$$\frac{d}{dt} \langle \mu, \mu \rangle = -4 \left\langle P_\mu, (Ric_\mu)^{1,1}_{|\xi_\mu^\perp} \right\rangle.$$

On the the other Theorem 5.4 implies

$$\left\langle P_\mu, (Ric_\mu)^{1,1}_{|\xi_\mu^\perp} \right\rangle = 2 \langle P_\mu, P_\mu \rangle$$

and so

$$\frac{d}{dt} \langle \mu, \mu \rangle = -8 \langle P_\mu, P_\mu \rangle \leq 0,$$

which readily implies that in the nilpotent case the unique solution to the system (4.7) is defined for every positive t . This fact together with Theorem 4.2 implies the statement of Theorem 1.1.

Moreover, we have the following

Proposition 5.7. *In the nilpotent SKT case the maximal solution to (4.7) converges to the the abelian bracket.*

Proof. Let $\mu(t)$ be the maximal solution to (4.7). Then, since $\|\mu(t)\|$ decreases, $\mu(t)$ converges to a bracket μ which is still 2-step nilpotent and SKT. Now, $(\rho_\mu^B)^{1,1}$ has to be zero since, otherwise, we should extend $\mu(t)$ after μ in a non-trivial way. Therefore $(\mathbb{R}^{2n}, J_0, \omega_0, \mu)$ is an SKT nilpotent Lie algebra having $(\rho_\mu^B)^{1,1} = 0$. Hence μ is the abelian bracket in view of a result of [4]. \square

Example 5.8. Here we study the basic Example 3.11 in terms of bracket flow. The starting bracket is

$$\mu_0 = -\frac{1}{2} \zeta^1 \wedge \zeta^{\bar{1}} \otimes Z_2 + \frac{1}{2} \zeta^1 \wedge \zeta^{\bar{1}} \otimes Z_{\bar{2}}$$

which corresponds to the bracket of the Lie algebra $\mathfrak{h}_3 \oplus \mathbb{R}$. Since the bracket flow preserves the center, we seek for a solution μ to (4.7) taking value only at $(Z_1, Z_{\bar{1}})$, i.e.

$$\mu = \mu_{1\bar{1}}^2 \zeta^1 \wedge \zeta^{\bar{1}} \otimes Z_2 + \mu_{1\bar{1}}^{\bar{2}} \zeta^1 \wedge \zeta^{\bar{1}} \otimes Z_{\bar{2}}.$$

For such a bracket we have

$$\rho_\mu^B = -2i |\mu_{1\bar{1}}^2|^2 \zeta^1 \wedge \zeta^{\bar{1}}$$

and

$$P_\mu = -2 |\mu_{1\bar{1}}^2|^2 \zeta^1 \otimes Z_1 + 2 |\mu_{1\bar{1}}^2|^2 \zeta^{\bar{1}} \otimes Z_{\bar{1}}.$$

Therefore

$$\delta_\mu(P_\mu)(Z_1, Z_{\bar{1}}) = 2\mu(P_\mu(Z_1), Z_{\bar{1}}) = -4 |\mu_{1\bar{1}}^2|^2 \mu(Z_1, Z_{\bar{1}})$$

and the corresponding bracket flow equation is

$$(5.3) \quad \begin{cases} \dot{z} = -2|z|^2 z, \\ z(0) = -\frac{1}{2} \end{cases}$$

where $z = \mu_{1\bar{1}}^2$. Since (5.3) has as solution the real map

$$z(t) = -\frac{1}{2(t+1)^{\frac{1}{2}}}$$

the solution $\mu(t)$ of the bracket flow is defined for every positive t and converges in \mathcal{A} to the null bracket corresponding to the abelian Lie algebra.

6. EVOLUTION OF TAMED SYMPLECTIC FORMS ON A COMPLEX MANIFOLD

Let (M, J) be a complex manifold. We recall that a symplectic form Ω on M *tames* J if

$$(6.1) \quad \Omega(JX, X) > 0$$

for every non-zero tangent vector field X on M . Such a condition is weaker than the compatibility of Ω with J since in this case the positive tensor induced by (6.1) is not symmetric. A structure (J, Ω) composed by a complex structure and a taming symplectic form was called in [18] a *Hermitian-symplectic* structure. Such a structure arises considering static solutions of the pluriclosed flow (1.3). Indeed if an SKT form ω satisfies the Hermitian-Einstein equation $r\omega = (\rho^B)^{1,1}(\omega)$ with $r \in \mathbb{R}$ and $r \neq 0$, then $\Omega = \frac{1}{r}\rho^B$ is a symplectic form taming J .

In [6] it was observed that Hermitian-symplectic structures are actually special SKT structures. This is because given a symplectic form Ω taming J and considered the decomposition of Ω in complex-type forms

$$\Omega = \omega + \beta + \bar{\beta} \in \Lambda^{1,1} \oplus \Lambda^{2,0} \oplus \Lambda^{0,2}$$

one has that $d\Omega$ vanishes if and only if β solves

$$(6.2) \quad \begin{cases} \bar{\partial}\Omega^{1,1} = -\partial\beta \\ \bar{\partial}\beta = 0. \end{cases}$$

In the sequel of the paper we are going to take into account the following evolution equation

$$(6.3) \quad \begin{cases} \frac{d}{dt}\Omega = -\rho^B(\omega) \\ \Omega(0) = \Omega_0, \end{cases}$$

which we will call the *Hermitian-symplectic (or simply HS) flow*.

Proposition 6.1. *Let Ω_0 be a tamed symplectic form on a compact complex manifold (M, J) . Then short-time existence of a solution $\Omega(t)$ of (6.3) is guaranteed. Moreover, $\Omega(t)$ is a symplectic form taming J for every t .*

Proof. We can write $\Omega_0 = \omega_0 + \beta_0 + \bar{\beta}_0$ and the Hermitian-symplectic flow decomposes in its $(1, 1)$ -part

$$(6.4) \quad \begin{cases} \frac{d}{dt}\omega = -(\rho^B)^{1,1}(\omega) \\ \omega(0) = \omega_0 \end{cases}$$

and the $(2, 0)$ -part

$$(6.5) \quad \begin{cases} \frac{d}{dt}\beta = -(\rho^B)^{2,0}(\omega) \\ \beta(0) = \beta_0. \end{cases}$$

Since (6.6) is the “usual” pluriclosed flow, it admits a solution $\omega(t)$ defined in an interval $[0, \varepsilon)$, for ε small enough. On the other hand, since $(\rho^B)^{2,0}(\omega)$ does not depend on β ,

$$\beta(t) = \beta_0 + \int_0^t (\rho^B)^{2,0}(\omega)(s) ds$$

is a solution to (6.5) and $\Omega(t) := \omega(t) + \beta(t)$ provides a solution to (6.3) which is unique in view of the uniqueness of solutions to (6.6).

We finally observe that the taming condition is preserved by the flow. Indeed, $\omega(t)$ is positive since solution to the pluriclosed flow and $\Omega(t)$ is closed since

$$\frac{d}{dt}(d\Omega(t)) = d\left(\frac{d}{dt}\Omega(t)\right) = -d\rho^B = 0,$$

and then $d\Omega(t)$ is constant. \square

The previous result says that the pluriclosed flow preserves the Hermitian-symplectic condition. Indeed, a Hermitian-symplectic structure can be defined as an SKT structure (ω_0, J) together a solution β to (6.2). As a consequence of Proposition 6.1 we have that if an SKT form ω_0 admits a solution β_0 to (6.2), then the solution $\omega(t)$ to the pluriclosed flow with initial condition ω_0 has a solution $\beta(t)$ for every t .

We recall the following stability theorem for the Hermitian curvature flow (1.2) obtained by Streets and Tian

Theorem 6.2. ([19]) *Let (M, g, J) be a complex manifold with a Kähler-Einstein metric g and $c_1(M) < 0$ or $c_1(M) = 0$. Then there exists $\epsilon = \epsilon(g)$ so that if \tilde{g} is a J -Hermitian metric on M and $\|\tilde{g} - g\|_{C^\infty} < \epsilon$, then the solution to (1.2) with initial condition \tilde{g} exists for all time and converges exponentially to a Kähler-Einstein metric.*

Corollary 6.3. *In the hypothesis of Theorem 6.2; let $\tilde{\Omega}$ be a symplectic form on M taming J and such that $\|\tilde{g} - g\|_{C^\infty} < \epsilon$, where \tilde{g} is the Hermitian metric of $\tilde{\Omega}^{1,1}$. Then the solution $\Omega(t)$ of flow (6.3) is defined for every $t \in [0, \infty)$ and it converges to symplectic form whose $(1, 1)$ -component induces a Kähler-Einstein metric.*

Proof. Let $\tilde{\omega} = \tilde{\Omega}^{1,1}$. Then using Theorem 6.2 we have that equation

$$(6.6) \quad \begin{cases} \frac{d\omega}{dt} = -(\rho^B)^{1,1}(\omega) \\ \omega(0) = \tilde{\Omega}^{1,1} \end{cases}$$

has a unique solution $\omega(t)$ defined in $[0, \infty)$ and converging to a Kähler-Einstein structure ω_∞ . Since $\omega(t)$ is defined in $[0, \infty)$, the system

$$\begin{cases} \frac{d\beta}{dt} = -(\rho^B)^{2,0}(\omega) \\ \beta(0) = (\tilde{\Omega})^{2,0}. \end{cases}$$

as a solution $\beta(t)$ in $[0, \infty)$ which can be written as

$$\beta(t) = \int_0^t f(s) ds + \beta(0);$$

$f(s)$ being

$$f(s) = -(\rho^B)^{2,0}(\omega(s)).$$

Now, since the metric induced by $\omega(t)$ converges exponentially to a Kähler-Einstein metric h , then f converges exponentially to zero. Therefore

$$\|f(s)\| \leq Ce^{-\lambda s},$$

for some positive constants C, λ . That implies

$$\lim_{t \rightarrow \infty} \|\beta(t) - \beta(0)\|_{C^\infty} = \left\| \int_0^\infty f(s) ds \right\|_{C^\infty} \leq C \int_0^\infty e^{-\lambda s} ds \leq \frac{C}{\lambda},$$

showing that $\beta(t)$ converges.

Therefore the flow (6.3) has a unique solution $\Omega(t)$ in $[0, \infty)$ converging to a form

$$\Omega_\infty = \omega(h) + \beta_\infty + \bar{\beta}_\infty.$$

Since $\beta(t)$ satisfies

$$\begin{cases} \bar{\partial}\beta(t) = \partial\omega(t) \\ \partial\beta(t) = 0, \end{cases}$$

$\beta(t)$ converges to a closed form β_∞ . Since the $(1, 1)$ component of Ω_∞ is a Kähler form with respect J , then Ω_∞ is a symplectic form.

Remark 6.4. Generically we do not expect that $\beta(t)$ converges to zero. A trivial counterexample is the following:

consider the standard complex torus $\mathbb{T}^{2n} = \mathbb{C}/\mathbb{R}^{2n}$ with the standard flat Kähler structure $\omega_0 = \sum dz^r \wedge d\bar{z}^r$. Then $\Omega_0 = \omega + dz^1 \wedge dz^2 + d\bar{z}^1 \wedge d\bar{z}^2$ is a Hermitian-symplectic structure and $\Omega(t) \equiv \Omega_0$ solves the flow (6.3).

□

6.1. Flow (6.3) on Lie algebras. Let (\mathfrak{g}, μ) be a Lie algebra endowed with a complex structure J . Let $\{Z_r\}$ an arbitrary $(1, 0)$ -frame with dual frame $\{\zeta^k\}$. Every Hermitian inner product g on (\mathfrak{g}, μ, J) can be written as

$$g = g_{r\bar{k}} \zeta^r \bar{\zeta}^{\bar{k}},$$

for some real coefficients $(g_{r\bar{k}})$. The inner product g induces the fundamental form

$$\omega = i g_{r\bar{k}} \zeta^r \wedge \bar{\zeta}^{\bar{k}}.$$

Therefore an arbitrary non-degenerate 2-form Ω dominating J writes as

$$\Omega = i g_{r\bar{k}} \zeta^r \wedge \bar{\zeta}^{\bar{k}} + \beta_{ij} \zeta^i \wedge \zeta^j + \bar{\beta}_{\bar{i}\bar{j}} \bar{\zeta}^{\bar{i}} \wedge \bar{\zeta}^{\bar{j}}$$

Using equations (3.6), we get that the problem (6.3) is equivalent to the following system

$$(6.7) \quad \begin{cases} \frac{d}{dt} g_{i\bar{j}} = -B_{i\bar{j}}^a B_{a\bar{r}}^r + B_{i\bar{j}}^a g^{r\bar{k}} \bar{B}_{r\bar{k}}^{\bar{l}} g_{a\bar{l}} + B_{i\bar{j}}^{\bar{b}} B_{\bar{b}\bar{r}}^{\bar{r}} - B_{i\bar{j}}^{\bar{b}} g^{k\bar{r}} B_{k\bar{r}}^l g_{l\bar{b}} \\ \frac{d}{dt} \beta_{ij} = -i B_{ij}^a B_{a\bar{r}}^r + i B_{ij}^a g^{r\bar{k}} \bar{B}_{r\bar{k}}^{\bar{l}} g_{a\bar{l}} + i B_{ij}^{\bar{b}} B_{\bar{b}\bar{r}}^{\bar{r}} - i B_{ij}^{\bar{b}} g^{k\bar{r}} B_{k\bar{r}}^l g_{l\bar{b}} \\ g_{i\bar{j}}(0) = h_{i\bar{j}} \\ \beta_{ij}(0) = h_{ij} \end{cases}$$

where

$$\Omega_0 = i h_{i\bar{j}} \zeta^i \wedge \bar{\zeta}^{\bar{j}} + h_{rs} \zeta^r \wedge \zeta^s + \bar{h}_{lm} \bar{\zeta}^{\bar{l}} \wedge \bar{\zeta}^{\bar{m}}$$

is the starting symplectic form taming J and

$$\Omega = i g_{i\bar{j}} \zeta^i \wedge \bar{\zeta}^{\bar{j}} + \beta_{rs} \zeta^r \wedge \zeta^s + \bar{\beta}_{lm} \bar{\zeta}^{\bar{l}} \wedge \bar{\zeta}^{\bar{m}}$$

is the solution to (6.3).

In real dimension four, the equations (6.7) can be simplified writing every J -Hermitian inner product on \mathfrak{g} in matrix notation as

$$(g_{i\bar{j}}) = \begin{pmatrix} x & z \\ \bar{z} & y \end{pmatrix}$$

where x, y are positive real numbers and $z \in \mathbb{C}$ satisfying

$$xy - |z|^2 > 0.$$

In this way the inverse of g is

$$(g^{\bar{j}i}) = \frac{1}{xy - |z|^2} \begin{pmatrix} y & -z \\ -\bar{z} & x \end{pmatrix}$$

Example 6.5. Consider the solvable Lie algebra \mathfrak{g} with structure equations

$$(24, -14, 0, 0)$$

endowed with the complex structure

$$J(e_1) = e_2, \quad J(e_3) = e_4.$$

Let $\{Z_1, Z_2\}$ be the $(1, 0)$ -frame

$$Z_1 = \frac{1}{2}(e_1 - ie_2), \quad Z_2 = \frac{1}{2}(e_3 - ie_4);$$

then

$$[Z_1, Z_{\bar{1}}] = [Z_2, Z_{\bar{2}}] = 0, \quad [Z_1, Z_{\bar{2}}] = -\frac{1}{2}Z_1, \quad [Z_{\bar{1}}, Z_2] = -\frac{1}{2}Z_{\bar{1}}, \quad [Z_1, Z_2] = \frac{1}{2}Z_1.$$

Consider the $(1, 0)$ -coframe

$$\zeta^1 = e^1 + ie^2, \quad \zeta^2 = e^3 + ie^4$$

dual to $\{Z_1, Z_2\}$. Then

$$d\zeta^1 = -i\zeta^1 \wedge e^4, \quad d\zeta^2 = 0.$$

There follows

$$d\zeta^1 = -\frac{1}{2}\zeta^{12} + \frac{1}{2}\zeta^{1\bar{2}},$$

i.e.

$$\partial\zeta^1 = -\frac{1}{2}\zeta^{12}, \quad \bar{\partial}\zeta^1 = \frac{1}{2}\zeta^{1\bar{2}}.$$

The generic symplectic form taming the fixed J is

$$\tilde{\Omega} = ix^2\zeta^{1\bar{1}} + iy^2\zeta^{2\bar{2}} + iz\zeta^{1\bar{2}} + i\bar{z}\zeta^{\bar{1}2} + w\zeta^{12} + \bar{w}\zeta^{\bar{1}\bar{2}}$$

where $x, y \in \mathbb{R}$, $z, w \in \mathbb{C}$ satisfy

$$x^2y^2 - |z|^2 > 0, \quad w = -z.$$

The Bismut Ricci form with respect to the Hermitian metric defined by $\Omega^{1,1}$ is then given by

$$\rho^B(\tilde{\omega}) = -\frac{zx^2}{4(x^2y^2 - |z|^2)}\zeta^{12} + \frac{zx^2}{4(x^2y^2 - |z|^2)}\zeta^{1\bar{2}} - \frac{\bar{z}x^2}{4(x^2y^2 - |z|^2)}\zeta^{\bar{1}2} + \frac{\bar{z}x^2}{4(x^2y^2 - |z|^2)}\zeta^{\bar{1}\bar{2}}$$

The HS flow reduces to

$$\begin{cases} \dot{x} = \dot{y} = 0, \\ \dot{z} = -\frac{zx^2}{4(x^2y^2 - |z|^2)} \\ \dot{w} = -\dot{z}, \end{cases}$$

with initial conditions $x(0) = x_0, y(0) = y_0, z(0) = z_0, w(0) = w_0$. Consider the equation $\dot{z} = -\frac{zx^2}{4(x^2y^2 - |z|^2)}$. Such an equation is radial in the sense that its solutions $z = \rho e^{i\theta}$ have θ constant. Therefore it can be rewritten as $\dot{\rho} = -\frac{\rho x^2}{4(x^2y^2 - \rho^2)}$ in terms

of the unknown real map ρ . Since $\dot{\rho} < 0$, ρ is decreasing unless ρ vanishes. This ensures that ρ tends to zero (and then z tends to zero). Therefore we have

$$\begin{cases} z_{\infty} = 0 \\ w_{\infty} = 0, \end{cases}$$

and thus

$$\Omega_{\infty} = ix_0^2 \zeta^{1\bar{1}} + iy_0^2 \zeta^{2\bar{2}}.$$

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